U.G. 3rd Semester Examination - 2019

MATHEMATICS

[HONOURS]

Course Code : MATH(H)CC-05-T

Full Marks : 60 Time : $2\frac{1}{2}$ Hours

The figures in the right-hand margin indicate marks.

Candidates are required to give their answers in their own words as far as practicable.

Symbols and notations have their usual meanings.

1. Answer any **ten** questions: 2×10=20

 i) Use sequential criterion for limits to show that the following limit does not exist

$$\lim_{x\to 0}\frac{1}{x}\sin\frac{1}{x}.$$

 ii) Give example of function f and g which are not continous at a point $c \in \mathbb{R}$ but the sum ftg is continous as c.

 iii) Using ε-δ definition, show that

$$\lim_{x\to\infty}\frac{[x]}{x}=1.$$

[Turn over]

iv) Verify what her ($\mathbb{R}$, d) is a metric space, where $d(x, y) = |x^2 - y^2|$, $\forall x, y \in \mathbb{R}$.

v) Define diameter of a set in a metric space (x, d).

vi) Does $f'(c) = 0$ always imply existence of an extremum of f at c? Justify.

vii) Give an example of a function which has a jump discontinuity in its domain of definition.

viii) Show that the equation $f(x)=xe^3-2$ has a root in [0, 1].

ix) Expand log sin (x+h) in power of h by Taylor's Theorem.

x) Give geometrical interpretation of Lagrange's Mean Value Theorem.

xi) Define limit point of a set in a metric space (x, d). Give one example.

xii) For a metric space X, show that a point $a \in X$ is a cluster point of $A \subset X$ if there exists $\{a_n\}_{n=1}^{\infty}$ in A such that $\underset{n\to\infty}{\text{Lt}}\, a_n = a$.

xiii) Show that there does not exist a function ϕ such that $\phi'(x) = f(x)$, where $f(x)=x-[x]$, $x \in [0, 2]$.

xiv) Discuss the applicability of Rolle's theorem for

$$f(x) = 2 + (x-1)^{\frac{2}{3}} \text{ in } [0, 2].$$

xv) Show that $f(x)=|x+2|$ is continuous at $x=-2$ but not differentiable at this point.

2. Answer any **four** questions: 5×4=20

i) Let f be a continuous function on [a, b] and c be any real number between f(a) and f(b), then show that there exist a real number x in (a, b) such that f(x)=c.

Construct an example to show that continuity of f is not necessary for the existence of such x as above. 3+2

ii) State and prove Rolle's theorem.

iii) Find the maxima and minima of the function

$$f(x) = \sin x + \frac{1}{2}\sin 2x + \frac{1}{3}\sin 3x \text{ for all } x \in [0, \pi].$$

iv) Define a Metric space show that $(\mathbb{R}^2, d)$ is a metric space, where the metric $d: \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}$ is defined by $d(x, y) = |x_1 - y_1| + |x_2 - y_2|$; $x, y \in \mathbb{R}^2$ when $x=(x_1, x_2)$, $y=(y_1, y_2)$.

v) Suppose n-th derivative of a function f exists finitely in a closed interval [a, a+h]. Then show that there exists a positive proper fraction θ satisfying the relation

$$f(a+h) = f(a) + hf'(a) + \frac{h^2}{2!}f''(a) + \ldots$$

$$+ \frac{h^{n-1}}{(n-1)!}f^{n-1}(a) + \frac{h^n}{n!}f^n(a+\theta h).$$

vi) Let f: $[a,b] \to \mathbb{R}$ be such that f has a local extremum as an interior point c of [a,b]. If $f'(c)$ exists, then prove that $f'(c)=0$.

Answer any **two** questions from **Question No. 3** to **Question No. 6**: 10×2=20

3. a) A function $f:[0, 1] \to [0, 1]$ is continous on [0,1]. Prove that there exists a point c in [0,1] such that f(c)=c.

   b) Show that every uniformly continous function on an interval is continous on that interval, but the converse is not true.

   c) Prove that the function $f(x) = \frac{1}{x}$, $x \in (0, 1]$ is not uniformly continous on (0, 1].

   3+(2+2)+3

4. a) If f' and g' exist for all $x \in [a, b]$ and $g'(x) \neq 0 \ \forall \ x \in (a, b)$, then prove that for some $c \in (a, b)$, $\frac{f(c) - f(a)}{g(b) - g(c)} = \frac{f'(c)}{g'(c)}$.

   b) Obtain the Maclaurin's series expansion of $\log(1+x)$, $-1 < x \leq 1$.

c) Show that

$$x-\frac{x^2}{2}<\log(1+x)<x-\frac{x^2}{2(1+x)},\ x>0.$$

3+4+3

5. a) Prove that in a metric space every open ball is an open set and every closed ball is a closed set.

b) Define the following with example:

i) Subspace of a metric space

ii) Separable metric space (3+3)+(2+2)

6. a) Show that the function f on [0, 1] defined as

$$f(x)=\frac{1}{2^n} \text{ when } \frac{1}{2^{n+1}}<x\le\frac{1}{2^n},\ n=0,1,2,\ldots,$$

$f(0)=0$ is discontinuous at $\frac{1}{2}, \left(\frac{1}{2}\right)^2, \left(\frac{1}{2}\right)^3, \ldots$.

b) Show that $\lim_{x\to\infty} a^x.\sin\frac{b}{a^x}=\begin{cases}0 & \text{if } 0<a<1\\ b & \text{if } a>1\end{cases}$.

c) Prove that in a metric space (X, d), the interior of a set $A\subset X$ is the largest open subset of A.

3+3+4
