U.G. 2nd Semester Examination - 2019

MATHEMATICS

[HONOURS]

Course Code : MTMH/CC-T-04

Full Marks : 60 Time : $2\frac{1}{2}$ Hours

The figures in the right-hand margin indicate marks.

The symbols and notations have their usual meanings.

1. Answer any **ten** questions: 2×10=20

a) Determine whether x=0 is an ordinary point or a regular singular point of the differential equation:

$$2x^2\left(\frac{d^2y}{dx^2}\right)+7x(x+1)\left(\frac{dy}{dx}\right)-3y=0.$$

b) Write down the Cauchy-Euler type of equation in connection with homogeneous linear differential equation.

[Turn Over]

c) Show that e^x and e^{3x} are solutions of the differential equation:

$$\frac{d^2y}{dx^2}-4\frac{dy}{dx}+3y=0$$

Are they independent? 1+1

d) Prove that if $f_1(x)$ and $f_2(x)$ are two solutions of

$$P_0(x)\frac{d^2y}{dx^2}+P_1(x)\frac{dy}{dx}+P_2(x)y=0$$

then $Af_1(x) + Bf_2(x)$ is also a solution of this equation where A and B are arbitrary constants.

e) Show that $f(x, y) = xy^2$ satisfies the Lipschitz condition on the rectangle $R : |x| \le 1, |y| \le 1$ but does not satisfy a Lipschitz condition on the strip $S : |x| \le 1, |y| < \infty$.

f) Show that for the problem $\frac{dy}{dx} = y, y(0) = 1$, the constant 'a' in Picard's theorem must be smaller than unity.

g) From definition prove that the four functions $3e^x, -4e^x, 5e^x$ and $6e^x$ are linearly dependent.

h) If S is defined by the rectangle $|x| \le a, |y| \le b$, then show that the function $f(x, y) = x \sin y + y \cos x$, satisfy the Lipschitz condition. Find the Lipschitz constant.

i) Prove that : $\left[\vec{a}\times\vec{b},\ \vec{b}\times\vec{c},\ \vec{c}\times\vec{a}\right]=\left[\vec{a}\,\vec{b}\,\vec{c}\right]^2$.

j) Find the equation of a plane which contains the straight line $\vec{r} = t\vec{\alpha}$ and is perpendicular to the plane containing the straight lines $\vec{r} = t_1\vec{\beta}$ and $\vec{r} = t_2\vec{\gamma}$ where t, t_1, t_2 are scalars and $\vec{\alpha}, \vec{\beta}, \vec{\gamma}$ are given vectors and $\vec{r}$ is the current vector.

k) Define a single-valued vector function of a scalar variable in a domain. Give an example. 1+1

l) If $\vec{r} = 3t\hat{i}+3t^2\hat{j}+2t^3\hat{k}$ then find $\frac{d\vec{r}}{dt}\times\frac{d^2\vec{r}}{dt^2}$.

m) Find div $\vec{F}$ and curl $\vec{F}$ when $\vec{F}$ = grad $(x^3+y^3+z^3-3xyz)$.

n) Evaluate $\int_1^2\left(\vec{r}\times\frac{d^2\vec{r}}{dt^2}\right)dt$ where $\vec{r} = 2t^2\hat{i}+t\hat{j}-3t^2\hat{k}$.

o) If the vectors $\vec{A}$ and $\vec{B}$ be irrotational, then show that the vector $\vec{A} \times \vec{B}$ is solenoidal.

2. Answer any **four** questions: 5×4=20

a) Solve by the method of variation of parameters the equation:

$$\frac{d^2y}{dx^2} + y = \sec^3 x \ \tan x.$$

b) Solve: $x^3 \frac{d^3y}{dx^3} + 2x^2 \frac{d^2y}{dx^2} + 2y = 10\left(x + \frac{1}{x}\right).$

c) Solve by the method of undetermined coefficients

$$\frac{d^2y}{dx^2} - 7\frac{dy}{dx} + 6y = (x-2)e^x.$$

d) Reduce the expression $(\vec{\beta}+\vec{\gamma}).\left[(\vec{\gamma}+\vec{\alpha})\times(\vec{\alpha}+\vec{\beta})\right]$ to its simplest form and prove that it vanishes when $\vec{\alpha}$, $\vec{\beta}$, $\vec{\gamma}$ are coplanar. 3+2

e) Prove that the necessary and sufficient condition for a vector $\vec{r} = \vec{f}(t)$ to have a constant direction is $\vec{f} \times \frac{d\vec{f}}{dt} = 0$.

f) Show that the vector $\vec{V} = (4xy - z^3)\hat{i} + 2x^2\hat{j} - 3xz^2\hat{k}$ is irrotational. Show that $\vec{V}$ can be expressed as the gradient of some scalar function ϕ. 2+3

3. Answer any **two** questions: 10×2=20

a) i) Obtain the power series solution of $y'' + (x-1)y' + y = 0$ in powers of $(x-2)$.

ii) Illustrate by an example that a continuous function may not satisfy a Lipschitz condition on a rectangle. Also give an example to show that the existence of partial derivative of f(x, y) is not necessary for f(x, y) to be a Lipschitz function. 5+5

b) i) Solve: $\frac{dx}{dt} - 7x + y = 0$

$$\frac{dy}{dt} - 2x - 5y = 0.$$

ii) For the differential equation

$$(x^2+1)\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} + 2y = 0, \text{ given } y=x \text{ is}$$

a solution. Reduce the order of the differential equation. Hence obtain another solution which is independent with the given one. Hence write the general solution. 5+5

c) If $\phi = 3x^2yz$, $\vec{F} = y\hat{i} - xz\hat{j} + x^2\hat{k}$, and C be the curve $x = t, y = 2t^2, z = t^3$ from $t = 0$ to $t = 1$, then evaluate the integrals

i) $\int_C \phi d\vec{r}$ and

ii) $\int_C \vec{F} \times d\vec{r}$. Also find the circulation of $\vec{F}$ round the curve C, where $\vec{F} = y\hat{i} + z\hat{j} + x\hat{k}$ and C is the circle $x^2+y^2=1$, $z=0$. 7+3

d) i) Prove that the necessary and sufficient condition that the vector field defined by the vector point function $\vec{F}$ with continuous derivatives be conservative is that curl $\vec{F} = \nabla \times \vec{F} = 0$.

ii) Evaluate: $\iint_S \vec{F}.\vec{n}\, dS$ where $\vec{F} = 6z\hat{i} - 4\hat{j} + y\hat{k}$ and S is that part of the plane $2x+6y+3z=10$ which is located in the first octant. 5+5
