

U.G. 1st Semester Examination - 2019

MATHEMATICS

[HONOURS]

Course Code : MATH(H)/CC-2-T

Full Marks : 60

Time : $2\frac{1}{2}$ Hours*The figures in the right-hand margin indicate marks.**Symbols have their usual meanings.*1. Answer any **ten** questions: $2 \times 10 = 20$ i) Find the value of $(1+i\sqrt{3})^{30}$.ii) If $x + \frac{1}{x} = 2\cos\frac{\pi}{7}$, then find the value of

$$x^7 + \frac{1}{x^7}.$$

iii) Find the remainder when $3x^4 - 4x^3 + 2x^2 - 9x + 1$ is divided by $2x + 1$.iv) Find the value of $(4 + 3\sqrt{-20})^{\frac{1}{2}} + (4 - 3\sqrt{-20})^{\frac{1}{2}}$.*[Turn over]*

v) If α , β and γ are the roots of $x^3+px+q=0$,

then find $\sum \frac{1}{\alpha^2 - \beta\gamma}$.

vi) If the roots of the equation $x^3-3x^2+ax+b=0$ are in A.P., then find the value of $a+b$.

vii) If $f, g: \mathbb{R} \rightarrow \mathbb{R}$ are two functions such that $f(x)=2x+1$ and $g(x)=x^2-2$, then find $g \circ f$.

viii) Find the inverse of the matrix:

$$\begin{pmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{pmatrix}.$$

ix) Find the equation whose roots exceed by 2 from the roots of the equation

$$4x^4+32x^3+83x^2+76x+21=0.$$

x) If a, b, c are all positive real numbers such that $a+b+c=1$, then prove that

$$\frac{1}{1-a} + \frac{1}{1-b} + \frac{1}{1-c} \geq \frac{9}{2}.$$

xi) For what value of k , the vectors $(k, 1, 0)$, $(1, k, 1)$ and $(0, 1, k)$ in $\mathbb{R}^3$ are linearly dependent?

xii) Show that 5^n-1 is divisible by 4 for all positive integer n .

xiii) Give an example of a mapping which is

a) One-to-one but not onto

b) Onto but not one-to-one.

xiv) State Descartes's rule of signs.

xv) Is the matrix multiplication commutative, in general? Justify.

2. Answer any **four** questions: $5 \times 4 = 20$

a) Determine the rank of the following matrix for different values of λ :

$$\begin{pmatrix} \lambda & 1 & 0 \\ 3 & \lambda-2 & 1 \\ 3(\lambda+1) & 0 & \lambda+1 \end{pmatrix}$$

b) If $x^4-14x^2+24x-k=0$ has four real and unequal roots, prove that k must lie between 8 and 11.

c) For a suitable h apply the transformation $x=y+h$ to remove the term containing x^2 from the equation

$$x^3-15x^2-33x+847=0$$

and then solve the transformed equation by Cardan's method. Hence find the roots of the equation.

d) Verify Cayley-Hamilton theorem for the matrix

$$A = \begin{pmatrix} 0 & 0 & 1 \\ 3 & 1 & 0 \\ -2 & 1 & 4 \end{pmatrix}$$

and hence find A^{-1} .

e) Prove that the necessary and sufficient condition for a mapping to be invertible is that it is one-to-one and onto.

f) If a and b are two integers with $b > 0$, then prove that there exists unique integers q and r such that $a = bq + r$ with $0 \leq r < b$.

g) Discuss consistency and solutions of the following system of linear equations:

$$x + ay + az = 1$$

$$ax + y + 2az = -4$$

$$ax - ay + 4z = 2$$

for different values of a .

Answer any two questions from Q. No. 3 to Q. No. 6:

$$10 \times 2 = 20$$

3. a) If $\cos \theta = \frac{1}{2} \left(a + \frac{1}{a} \right)$ and $\cos \phi = \frac{1}{2} \left(b + \frac{1}{b} \right)$, then

show that $\cos(\theta + \phi)$ is one of the values of

$$\frac{1}{2} \left(ab + \frac{1}{ab} \right).$$

b) If a, b, c, d are positive reals such that $a+b+c+d=1$ then prove that

$$\frac{a}{1+b+c+d} + \frac{b}{1+a+c+d} + \frac{c}{1+a+b+d} + \frac{d}{1+a+b+c} \geq \frac{4}{7}.$$

c) Expand $\cos^7 \theta$ in a series of cosines of multiple of θ .

d) Show that

$$W = \{(x, y, z) : x \geq 0, x, y, z \in \mathbb{R}\}$$

is not a subspace of $\mathbb{R}^3$.

$$3+3+2+2$$

4. a) Let R be an equivalence relation on a set S and for $a \in S$, let $[a]$ denote the R -equivalence class of a in S . For any two elements $x, y \in S$ if $[x] \neq [y]$, then show that $[x] \cap [y] = \phi$.

b) Prove that the product of any m consecutive integers is divisible by m .

c) Solve the biquadratic equation:

$$x^4 - 6x^2 + 16x - 15 = 0. \quad 3+3+4$$

5. a) State and prove Cayley-Hamilton Theorem.

b) Reduce the matrix A to row reduced echelon form and then find its rank

$$A = \begin{pmatrix} 0 & 1 & -3 & 1 \\ 1 & 0 & 1 & 1 \\ 3 & 1 & 0 & 2 \\ 1 & 1 & -2 & 0 \end{pmatrix}.$$

c) Prove that $1!3!5! \dots (2n-1)! > (n!)^n$.

d) If a, b, c are positive reals and $abc = k^3$, then prove that $(1+a)(1+b)(1+c) \geq (1+k)^3$.

$$3+3+2+2$$

6. a) If z_1 and z_2 be two complex numbers, then show that

$$\text{i) } |z_1 \pm z_2| \geq ||z_1| - |z_2||$$

$$\text{ii) } z_1 \bar{z}_2 + \bar{z}_1 z_2 \leq 2|z_1||z_2|$$

where $\bar{z}$ is the conjugate of z .

b) For any two non-empty sets X and Y , let $f: X \rightarrow Y$ be a mapping such that

$$f(A \cap B) = f(A) \cap f(B)$$

for all non-empty sets A and B of X . Prove that f is injective.

c) Show that the open intervals $(0, 1)$ and $(0, \infty)$ have the same cardinality. $(2+2)+3+3$